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Boussinesq plumes and jets with decreasing source strengths in stratified environments

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Solutions to the equations of Morton *et al.* (*Proc. R. Soc. Lond.* A, vol. 234, 1956, p. 1) describing turbulent plumes and jets rising in uniformly stratified environments are identified for the first time.

The evolution of plumes and jets with sources whose driving flux decreases with time is considered in a stratified environment. Numerical calculations indicate that as the source buoyancy flux, for a Boussinesq plume (or source momentum flux, for a Boussinesq jet), is decreased, a transitional narrowing region with characteristic spreading angle $\tan^{-1}(2\alpha/3)$ is formed, where α is the well-known entrainment coefficient. The plume or jet dynamics are modelled well by a separable solution to the governing equations which predicts stalling in the plume at a critical stall time $t_s = \pi/N$ and stalling in the jet at a critical stall time $t_s = \pi/(2N)$, where N is the buoyancy frequency of the ambient background stratification. This stall time is independent of the driving source conditions, a prediction which is verified by numerical solution of the underlying evolution equations.

1. Introduction

Plumes and jets are ubiquitous throughout a large range of flows. Their abundance in geophysical flows is well-known and examples range from large-scale meteorological plumes rising over the desert to explosive volcanic eruptions or saline plumes descending from melting sea ice. In most large-scale geophysical flows, both the stratification of the ambient background fluid and the temporally varying nature of the source conditions are of crucial importance.

Plumes and jets with temporally varying source conditions in a homogeneous environment were considered in Scase *et al.* (2006) (referred to herein as S06), motivated by the very successful and popular earlier work of Morton, Taylor & Turner (1956), Zeldovich (1937) and more recently Hunt *et al.* (2003). In S06 it was demonstrated that when the driving source conditions (i.e. source of buoyancy or momentum) of a given plume or jet are reduced in an unstratified ambient, a narrowing region is found. This narrowing region is described well by a class of separable solutions to the governing equations. However, the identified narrowing region had a non-zero minimum radius away from the origin and so no pinch-off into separate puffs was predicted. This demonstrated the robust nature of plumes in unstratified ambient fluids to changes in their source conditions.

In the present paper we shall focus on Boussinesq plumes and jets only. We begin in § 2 by briefly reviewing the time-dependent governing equations in a stratified ambient background fluid. In the conventional time-independent case with constant buoyancy frequency we derive previously overlooked analytical series solutions which predict well the maximum height of rise of both plumes and jets, i.e. the height at which the momentum flux first drops to zero. We do not consider the final height to which the plume or jet collapses. This final collapse height is always below the maximum height of rise and is the height at which the fluid is neutrally buoyant. We restrict our analysis to the maximum rise height since the complicated entrainment processes that occur during the collapse phase between the maximum rise height and the final collapse height are beyond the scope of our model equations (see Bloomfield & Kerr 2000 for further discussion).

In § 3, we investigate the extent to which reduction in source strength affects the plume evolution and height of rise. We calculate numerically the temporal evolution of the bulk properties of a Boussinesq plume rising through a stratification where the plume is subject to a reduction in its driving buoyancy flux, using the equations derived in S06 and the steady plume solution derived in § 2 as the appropriate initial condition. We use the results of these numerical calculations to motivate the theoretical analysis of § 4, where we identify a class of attracting separable solutions, equivalent to those found in homogeneous ambient fluids in S06. These solutions provide an intermediate asymptotic for all transient reductions in source strength, providing the source strength is reduced rapidly enough. We demonstrate close agreement between our theory and the numerical solutions, and in § 5 we draw our conclusions.

2. The system of governing equations

We consider a stratified ambient background fluid with buoyancy frequency, N, defined by $\rho_\infty(z) = \rho_0 \exp\{-N^2 z/g\}$, where $\rho_\infty(z)$ is the density of the ambient fluid, ρ_0 is a reference density equal to the density of the ambient fluid at the origin, z is the vertical distance from the origin and g is the acceleration due to gravity. The Boussinesq system of equations describing the system, derived in S06, is

$$\frac{\partial}{\partial t}\left(\frac{Q^2}{M}\right) + \frac{\partial Q}{\partial z} = 2\alpha\rho_0^{1/2}M^{1/2}, \quad \frac{\partial Q}{\partial t} + \frac{\partial M}{\partial z} = \frac{QF}{M}, \quad \frac{\partial}{\partial t}\left(\frac{QF}{M}\right) + \frac{\partial F}{\partial z} = -N^2 Q. \tag{2.1a–c}$$

The mass flux, buoyancy flux and momentum flux are defined respectively as $Q = b^2 w\rho$, $M = b^2 w^2 \rho$, $F = b^2 wg$, where b is the plume radius, w is the centreline velocity and ρ is the plume density. Equations (2.1) are derived under the assumptions that the plume has no radial variations in either velocity or density (i.e. they have top-hat profiles) and that entrainment into the plume can be modelled by a constant entrainment coefficient α (Morton *et al.* 1956; S06). We recall the steady solutions to the above system (2.1), with $N = 0$ (i.e. unstratified background fluid), due to Morton *et al.* (1956):

$$Q_u(z) = \frac{6\alpha}{5}\left(\frac{9\alpha}{10}\right)^{1/3} F_0^{1/3}\rho_0^{2/3}z^{5/3}, \quad M_u(z) = \left(\frac{9\alpha}{10}\right)^{2/3} F_0^{2/3}\rho_0^{1/3}z^{4/3}, \quad F_u(z) = F_0. \tag{2.2}$$

It is well-known that both plumes and jets rising through a stratification with constant buoyancy frequency N have finite maximum rise heights, z_{hp} and z_{hj} respectively, where the vertical velocity first decreases to zero, with (see e.g. Turner 1973)

$$z_{hp} \approx 2.572 z_{cp}, \quad z_{hj} \approx 1.429 z_{cj}, \quad \text{where} \quad z_{cp} = \left(\frac{F_0}{4\alpha^2\rho_0 N^3}\right)^{1/4}, \quad z_{cj}\left(\frac{M_0}{4\alpha^2\rho_0 N^2}\right)^{1/4}. \tag{2.3}$$

Importantly, the finite maximum rise heights scale with the quarter-power of the source buoyancy flux and momentum flux respectively (the constants of proportionality have been determined numerically).

2.1. *Steady Boussinesq plume*

We begin by considering the so-called steady 'pure' point-source plume with the various fluxes functions of position, z, alone. We take a constant source buoyancy flux $F_0 > 0$ and hence we have the boundary conditions $Q(0) = M(0) = 0$, $F(0) = F_0$. Dividing the time-independent forms of equations (2.1b) and (2.1c) and integrating the resulting equation for $\mathrm{d}M/\mathrm{d}F$ given the pure plume boundary conditions we obtain

$$M = \frac{1}{N}\left(F_0^2 - F^2\right)^{1/2}. \tag{2.4}$$

This demonstrates that $|F| \leqslant F_0$ for all valid z, i.e. that the maximum buoyancy flux for a plume must occur at its source, as expected. Combining the time-independent form of (2.1a) and (2.4) then yields

$$\frac{\mathrm{d}^2 F}{\mathrm{d}z^2} = -2\alpha\rho_0^{1/2} N^{3/2}\left(F_0^2 - F^2\right)^{1/4}. \tag{2.5}$$

Introducing the scaling $F = F_0\hat{F}$ and $z = z_{cp}\hat{z}$ allows (2.5) to be written as

$$\frac{\mathrm{d}^2 \hat{F}}{\mathrm{d}\hat{z}^2} = -(1 - \hat{F}^2)^{1/4}. \tag{2.6}$$

As $z_{cp} \to \infty$, the steady unstratified solutions (2.2) are recovered. All other quantities may be derived immediately once $\hat{F}$ is known.

Guided by the observation that if $N \ll 1$ then $Q \propto z^{5/3}$ and the steady form of (2.1c) implies that the first-order correction to F is of the functional form $z^{8/3}$, we seek a solution of (2.6) of the form

$$\hat{F}(\hat{z}) = \sum_{n=0}^{\infty} a_n \hat{z}^{8n/3}, \tag{2.7}$$

together with the boundary conditions that $\hat{F}(0) = 1$ and $\mathrm{d}\hat{F}/\mathrm{d}\hat{z}|_{\hat{z}=0} = 0$ (consistent with $Q(0) = 0$). It can be seen that if (2.6) is to contain a non-trivial plume-like solution then there must be at least two distinct solutions satisfying the nonlinear second-order differential equation (2.6) and the associated boundary conditions, since $\hat{F} = 1$ clearly satisfies all conditions and the governing equation.

By ensuring that the coefficient of each power of $\hat{z}$ balances when each side of (2.6) is raised to the fourth power, we obtain

$$a_0 = 1, \quad \left(\frac{40}{9}\right)^4 a_1^4 = -2a_1, \tag{2.8}$$

and for $n \geqslant 2$

$$\begin{aligned} a_n = {} & \left[\frac{8n\,(8n-3)}{5} - 2\right]^{-1} \left\{ 2^{1/3}\left(\frac{9}{10}\right)^{2/3} \sum_{k=2}^{n-1} c_{(k,n+1)} a_k a_{n-k+1} \right. \\ & \left. + \sum_{k=3}^{n}\left[\sum_{j=1}^{k-1} c_{(j,k)} a_j a_{k-j}\right]\left[\sum_{j=1}^{n+2-k} c_{(j,n+3-k)} a_j a_{n+3-k-j}\right] + \sum_{k=1}^{n-1} a_k a_{n-k} \right\}, \end{aligned} \tag{2.9}$$

where $c_{(j,k)} = (8/9)^2 j(8j-3)(k-j)[8(k-j)-3]$.

It can be seen from the equation for a_1 (2.8) that indeed there are two solutions of this form for the governing equation (2.6) with the required boundary conditions, given by the choice of either $a_1 = 0$, which leads to $\hat{F} = 1$, or $a_1 = -(9/20)^{4/3}/2$, which leads to the desired non-trivial plume solution. The resulting series appears, from numerical calculation, to converge rapidly.

2.2. *Steady Boussinesq forced plume*

We now turn to the case of a steady forced point-source plume, as originally considered by Morton (1959), where $Q(0) = 0$, $M(0) = M_0$ and $F(0) = F_0$ for some initial momentum and buoyancy fluxes $M_0 \neq 0$ and F_0 respectively. The analogue of (2.4) is now

$$M = \frac{1}{N}\left\{\left[N^2 M_0^2 + F_0^2\right] - F^2\right\}^{1/2}, \tag{2.10}$$

and we introduce non-dimensional quantities given by $F = \{N^2 M_0^2 + F_0^2\}^{1/2}\hat{F}$ and $z = z_{cj}[\{N^2 M_0^2 + F_0^2\}^{1/2}/(N M_0)]^{1/4}\hat{z}$, thus recovering (2.6) together with the boundary conditions $\hat{F}(0) = F_0/[N^2 M_0^2 + F_0^2]^{1/2}$, $\mathrm{d}\hat{F}/\mathrm{d}\hat{z}|_{\hat{z}=0} = 0$. We define a quantity $0 \leqslant \theta < 1$ such that $\hat{F}(0) = \theta$. The parameter θ is a measure of relative importance of source buoyancy and source momentum, with $\theta = 0$ being a pure jet and $\theta = 1$ being a pure plume. The solution in § 2.1 corresponds to $\theta = 1$, so we now assume that $0 \leqslant \theta < 1$.

As in the previous section, we note that for a steady jet in an unstratified ambient fluid, $Q \propto z$, hence a first-order correction to F for $N \ll 1$ would have functional form z^2; thus we now seek a solution of the form

$$\hat{F}(\hat{z}) = \sum_{n=0}^{\infty} a_n \hat{z}^{2n}. \tag{2.11}$$

Substitution of (2.11) into (2.6) and equating all powers of $\hat{z}$ gives

$$a_0 = \theta, \quad a_1 = -\frac{(1-\theta^2)^{1/4}}{2}, \tag{2.12}$$

with for $n \geqslant 2$

$$\begin{aligned} a_n = \left[8n(2n-1)(1-\theta^2)^{3/4}\right]^{-1} &\left\{ 2\left[1-\theta^2\right]^{1/2} \sum_{k=2}^{n-1} c_{(k,n+1)} a_k a_{n-k+1} \right. \\ &\left. + \sum_{k=3}^{n} \left[\sum_{j=1}^{k-1} c_{(j,k)} a_j a_{k-j}\right] \left[\sum_{j=1}^{n+2-k} c_{(j,n+3-k)} a_j a_{n-k-j+3}\right] + \sum_{k=0}^{n-1} a_k a_{n-k-1} \right\}, \end{aligned} \tag{2.13}$$

where $c_{(j,k)} = 4j(2j-1)(k-j)[2(k-j)-1]$.

When $\theta = 0$, we find $a_{2n} \equiv 0$ and numerical calculation of the ratio $a_{2n+3} z_{hj}/a_{2n+1}$ indicates again that the series converges rapidly. However, the series is not appropriate for large values of θ and in the limiting case of the pure plume, $\theta = 1$, the series solution (2.11)–(2.13) diverges for all positive $\hat{z}$, thus we are unable to recover (2.7)–(2.9).

Figure 1(*a*) is a comparison of the analytical solutions of (2.7)–(2.9) and (2.11)–(2.13) (solid) compared with numerical (dashed) solutions. The figure demonstrates that even though only five terms of the series for $\hat{F}$ have been used, the solution has converged quickly to the numerical solution. The theoretical solutions presented, using only five terms of the series solution, are indistinguishable from plots where 150 terms of the series solution have been used.

Taking the leading two terms of both (2.7) and (2.11) allows us to predict approximately the rise height of a pure plume, z_{hp}, and the rise height of a forced plume, z_{hj},

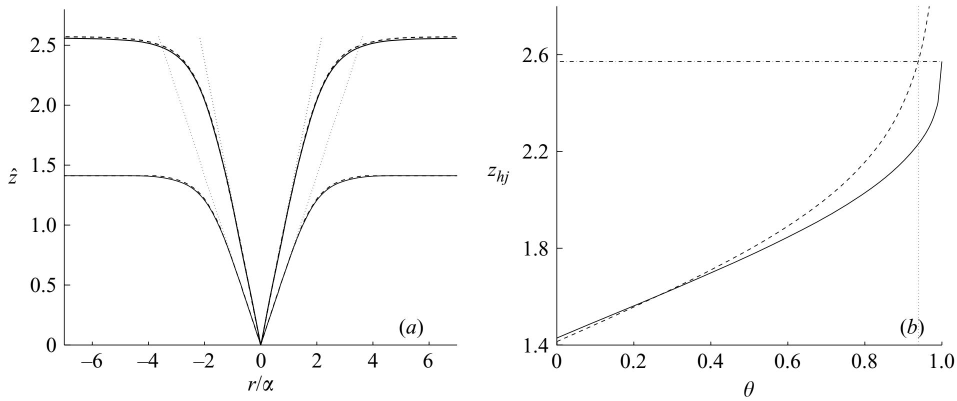

FIGURE 1. (*a*) A comparison of numerical solution (dashed) and the first 5 terms of the analytical solution of the steady stratified Boussinesq plume (solid bold) and pure jet ($\theta = 0$) (solid) equations. The unstratified solutions $b = 6\alpha z/5$ (dotted bold) and $b = 2\alpha z$ (dotted) are also shown. (*b*) A comparison of numerically calculated (solid) rise heights, z_{hj}, for a forced plume as θ varies and theoretical predictions (dashed) (2.14). For small to moderate values of θ the theoretical results agree well with the numerical results. However for $\theta \gtrsim 0.9391$ (dotted), z_{hj} is greater than the numerically calculated rise height for a pure plume source, $\theta = 1$ (dot-dashed).

as

$$z_{hp} \sim 2^{5/4}\left(\frac{10}{9}\right)^{1/2}\left[\frac{F_0}{4\alpha^2\rho_0 N^3}\right]^{1/4}, \quad z_{hj} \sim 2^{1/2}\left[\frac{(1+\theta)^3}{1-\theta}\right]^{1/8}\left\{\frac{\left[N^2M_0^2+F_0^2\right]^{1/2}}{4\alpha^2\rho_0 N^3}\right\}^{1/4}, \tag{2.14}$$

respectively. Numerical evaluation of the rise heights in (2.14) yields $z_{hp} \sim 2.507 z_{cp}$, and $z_{hj} \sim 1.414 z_{cj}$ (for $\theta = 0$) respectively, which are in good agreement with previous studies (cf. (2.3) where the solutions were calculated numerically).

Figure 1(*b*) is a comparison of numerically calculated rise heights (solid line) and the expression given in (2.14) (dashed line) for $0 \leqslant \theta < 1$. The graph demonstrates that the theoretical predictions based on only the first two terms of the series (2.11) agree well with the numerical prediction for small to moderate values of θ. For larger values of θ, z_{hj} over-predicts the rise height, indeed exceeding the numerically calculated rise height of a pure plume source ($\theta = 1$) when $\theta \gtrsim 0.9391$.

3. Numerical solutions

It was shown by Morton *et al.* (1956) that a turbulent pure point-source plume spreads as a cone of semi-angle $\tan^{-1}(6\alpha/5)$, independent of the strength of the driving source buoyancy flux. In S06 it was shown that a plume, or jet, propagating through an unstratified ambient fluid, which is subject to a reduction in source strength, develops a narrowing region which approaches a cone with semi-angle $\tan^{-1}(2\alpha/3)$. We now investigate the behaviour of such a flow when the ambient fluid is stratified. The numerical scheme employed is a simple extension of that presented in Appendix A of S06.†

† Equation (A2) of S06 is solved with an additional term, $-4^{1/3}(9/10)^{4/3}(z/z_{cp})^{8/3}\tilde{M}$, in the third row of the second matrix on the right hand side.

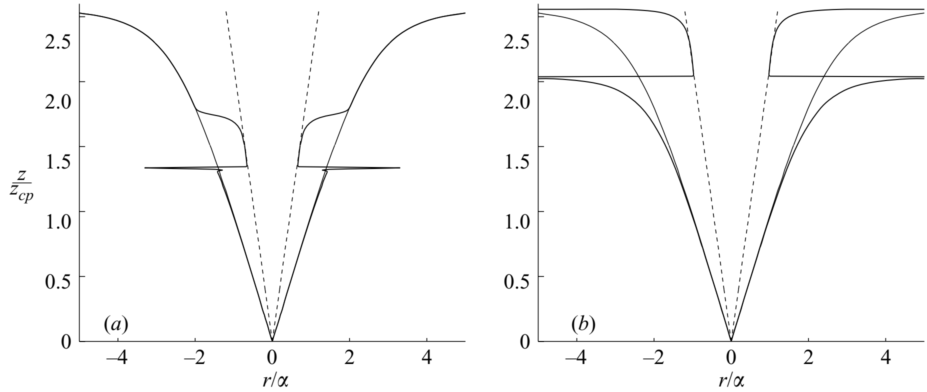

FIGURE 2. A numerical experiment showing the development of the characteristic radius for an established plume rising through a uniformly stratified ambient fluid ($N = 1\,\mathrm{s}^{-1}$), which has its driving source buoyancy flux reduced by 60%. In the simulation shown $\alpha = 1$. The initial plume profile is shown as a solid (thin) line, the narrowing region defined by $b = 2\alpha z/3$ is shown dotted. The solid bold line in (*a*) shows the profile at $t = 2\pi/(5N)$ s and in (*b*) at $t = \pi/N$ s.

We consider a plume with pure plume source conditions that has been established for all negative time, such that the quantities Q, M and F are as given in §2.1. At $t = 0$ the source buoyancy flux is reduced from F_0 to $0 \leqslant F_1 < F_0$, hence the boundary condition on the buoyancy flux is given by $F(0,t) = F_0 + (F_1 - F_0)H(t)$, where H is the Heaviside step function.

Figure 2 shows the temporally evolving flow of a plume due to the reduction in the driving buoyancy flux. 'New' plume fluid has a lower buoyancy flux associated with it than the 'old' plume fluid and so must have a smaller maximum rise height (2.3). Near the origin, both old and new plume fluid rise as if in an unstratified ambient with characteristic radius $b(z,t) \approx 6\alpha z/5$. It can be seen that for all times shown, the plume has three characteristic regions, just as in the unstratified case. Near the source, the plume behaves as if it has risen from a source with the final reduced buoyancy flux, while at greater heights the plume behaves as if it has risen from a source with the initial buoyancy flux. These two regions are connected by a transitional region, where the spreading angle is markedly reduced, and appears to approach $\tan^{-1}(2\alpha/3)$, just as in the unstratified case considered previously (S06).

4. Time-dependent separable solutions

4.1. *Plume solutions*

Motivated by the behaviour shown in figure 2, we seek a separable solution to (2.1) where

$$b^2 = \frac{Q^2}{M\rho_0} = \frac{4\alpha^2 z^2}{9}. \tag{4.1}$$

From (2.1*a*) it follows that $\partial Q/\partial z = 3Q/z$ and hence

$$Q = \frac{4\alpha^2\rho_0}{9} z^3 q_1(t), \qquad M = \frac{4\alpha^2\rho_0}{9} z^4 q_1(t)^2, \tag{4.2}$$

where $q_1(t)$ is a function of t alone. Equation (2.1b) implies that

$$F = \frac{4\alpha^2\rho_0}{9} z^4 \left[\frac{\mathrm{d}q_1}{\mathrm{d}t} + 4q_1^2\right] q_1, \tag{4.3}$$

and so substitution into (2.1c), implies that $q_1(t)$ satisfies

$$\frac{\mathrm{d}^2 q_1}{\mathrm{d}t^2} + 12 q_1 \frac{\mathrm{d}q_1}{\mathrm{d}t} + 16 q_1^3 + N^2 q_1 = 0. \tag{4.4}$$

This equation has stationary points for $q_1 = 0$, the trivial solution (which does not satisfy the boundary conditions), and also $q_1^2 = -N^2/16$, which has no real solutions unless $N^2 < 0$, i.e. unless we have a statically unstable background stratification. Letting $N^2 = -N_0^2 < 0$, a constant, shows that the stationary point is $q_1 = N_0/4$ and this yields $b = 2\alpha z/3$, $w = N_0 z/4$, i.e. the steady solutions identified by Batchelor (1954), and discussed in more detail in Caulfield & Woods (1998) and S06.

Any useful time-dependent solution of (4.4) must approach the time-dependent similarity solutions in an unstratified environment (cf. S06) as $N \to 0$. An exact solution to (4.4), which tends to the correct functional limit as $N \to 0$, is $q_1(t) = (N/4)\cot[(Nt)/2]$, which yields

$$Q = \frac{N\alpha^2\rho_0}{9} z^3 \cot\left(\frac{Nt}{2}\right), \quad M = \frac{N^2\alpha^2\rho_0}{36} z^4 \cot^2\left(\frac{Nt}{2}\right), \tag{4.5a, b}$$

$$F = \frac{N^3\alpha^2\rho_0}{72} z^4 \cot\left(\frac{Nt}{2}\right)\left\{\cot^2\left(\frac{Nt}{2}\right) - 1\right\}. \tag{4.5c}$$

The corresponding plume width, velocity and buoyancy force are therefore given by

$$b = \frac{2\alpha z}{3}, \quad w = \frac{Nz}{4}\cot\left(\frac{Nt}{2}\right), \quad g' = \frac{N^2 z}{8}\left[\cot^2\left(\frac{Nt}{2}\right) - 1\right]. \tag{4.6}$$

When $t = \pi/N$, $w = 0$ for all z where the above solution (4.5) is realized, and so we expect a region of the plume with non-zero vertical extent to stall at this time. Furthermore, at $t = \pi/(2N)$, $F = 0$ and $g' = 0$ over a region of the plume, i.e. this region does not experience any buoyancy forces acting upon it and propagates purely under its own momentum.

This argument is perhaps more clearly described in terms of the 'laziness parameter', $\Gamma(z, t)$ (e.g. Morton 1959; Hunt & Kaye 2005), taking care to note that $\Gamma(z, t)$ is not related to θ, defined in § 2.2. This non-dimensional parameter is defined, in the present notation, as

$$\Gamma(z, t) = \frac{5}{8\alpha\rho_0^{1/2}} \frac{Q^2 F}{M^{5/2}}, \tag{4.7}$$

and is a measure of how plume-like or jet-like the flow is. In a pure plume rising through an unstratified ambient $\Gamma(z, t) = 1$ everywhere, independent of z, while for a pure jet rising through an unstratified ambient $\Gamma(z, t) = 0$ everywhere. For a pure plume rising through a stratified environment, as $z \to z_{hp}$, the maximum rise height of the plume, $\Gamma(z) \to -\infty$. In the narrowing region of a plume rising through an unstratified ambient, it was shown in S06 that $\Gamma(z, t) = 5/6$. For the solution (4.5), we find

$$\Gamma(z, t) = \frac{5}{6}\left[1 - \tan^2\left(\frac{Nt}{2}\right)\right]. \tag{4.8}$$

It follows therefore that initially, $\Gamma(z,t)=5/6$, i.e. the narrowing region behaves as if the background fluid were unstratified, which we observe for a steady plume too ($b(z)\approx 6\alpha z/5$ near the origin). However, $\Gamma(z,t)$ decreases and at $t=\pi/(2N)$, $\Gamma(z,t)=0$ and the narrowing region becomes jet-like. This region experiences no buoyancy forces and its motion is due to its momentum alone. As $t\to\pi/N$, $\Gamma(z,t)\to-\infty$ and the narrowing region has equivalent laziness to a plume as it approaches its maximum rise height. Therefore over the first half of its evolution in time, this solution at a given height, z, loses its initial buoyancy. Over the second half of its evolution it loses its momentum, and hence stalls, as the buoyancy force now acts to decelerate the plume since it is dense relative to its surroundings. Interestingly, the stall time is independent of the initial source conditions.

4.2. *Jet solutions*

Since the spreading angle of the jet is observed to reduce to $\tan^{-1}(2\alpha/3)$, we follow the same procedure described above to attain (4.4). The exact solution of (4.4), which tends to the jet solutions identified in § 3.6 of S06 in the limit $N\to 0$, is

$$q_1(t)=\frac{N}{4}\cot(Nt). \tag{4.9}$$

This yields a mass flux, momentum flux and buoyancy flux given by

$$Q=\frac{N\alpha^2\rho_0}{9}z^3\cot(Nt),\quad M=\frac{N^2\alpha^2\rho_0}{36}z^4\cot^2(Nt),\quad F=-\frac{N^3\alpha^2\rho_0}{36}z^4\cot(Nt), \tag{4.10}$$

with corresponding characteristic jet radius, velocity and reduced gravity given by

$$b=\frac{2\alpha z}{3},\quad w=\frac{Nz}{4}\cot(Nt),\quad g'=-\frac{N^2 z}{4}. \tag{4.11}$$

Hence, when $t=\pi/(2N)$, $w=0$ for all z where the above solution (4.10) is realized, and so we expect a region of the jet with non-zero vertical extent to stall at this time. It is unsurprising that this is a shorter time than for the plume, as the fluid is never buoyant compared to the ambient. It should be noted that although the reduced gravity in this case is independent of time, the position of the narrowing region changes with time and so it should not be inferred that the narrowing region has a steady reduced gravity.

For the solution given by (4.10), we have a laziness parameter (4.7) defined by

$$\Gamma(z,t)=-\tfrac{5}{3}\tan^2(Nt). \tag{4.12}$$

Hence, initially the narrowing region behaves as a pure jet, but as $t\to\pi/(2N)$, is forced to stall with $\Gamma(z,t)\to-\infty$.

4.3. *Comparison with numerical method*

Figure 3 is a snapshot of the evolution of the buoyancy flux, F, at time $t=\pi/(2N)$ (bold) and $t=\pi/(4N)$ (solid), for the case $\alpha=1$, $N=1\,\mathrm{s}^{-1}$. Initially, at $t=0$, the buoyancy flux, F, lies on the dashed curve. The buoyancy flux at the origin is reduced from $F_0=1\,\mathrm{kg\,m\,s}^{-3}$ to $F_1=0.4\,\mathrm{kg\,m\,s}^{-3}$. Therefore the final curve that F must lie upon is the initial dashed curve rescaled in the F-direction by a factor of 0.4 and rescaled in the z-direction by a factor of $0.4^{1/4}$ (see (2.3)), whose profile is shown as a dot-dashed line. At the time shown, the upper part of the plume remains unaffected by changes at the source, since the information has not reached that far up the plume. The lower part of the plume has adjusted to the new buoyancy flux at the origin and is lying on the dot-dashed curve. At time $t=\pi/(2N)$, the narrowing region

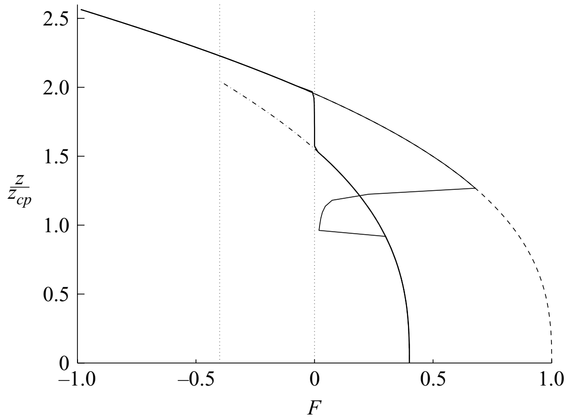

FIGURE 3. The buoyancy flux, F, at time $t = \pi/(2N)$ s (bold) and at time $t = \pi/(4N)$ s (solid) with $N = 1\,\mathrm{s}^{-1}$ and $\alpha = 1$ for a plume with buoyancy flux reduced from $F_0 = 1\,\mathrm{kg\,m\,s^{-3}}$ to $F_0 = 0.4\,\mathrm{kg\,m\,s^{-3}}$ at $t = 0$ s. The initial buoyancy flux profile is shown dashed, the final profile is shown dot-dashed. At time $t = \pi/(2N)$ s, the solution in (4.5) is in good agreement with the numerical solution shown (bold). A portion of the plume, corresponding to the narrowing region, has zero buoyancy flux ($F = 0$) and can only propagate as a result of its momentum. The minimum final buoyancy attained at the maximum of rise for the final reduced source buoyancy flux $F = -0.4\,\mathrm{kg\,m\,s^{-3}}$ is shown as a dotted vertical line.

corresponds to the vertical section joining the upper and lower parts of the bold line. The vertical dotted line $F = 0$ is the solution (4.5) evaluated at $t = \pi/(2N)$ s. It can be seen that the narrowing region agrees well with the predicted separable solution (4.5) and, in particular, that the buoyancy flux has dropped very close to zero over a finite vertical region (i.e. the density in the plume decreases linearly with height, exactly in step with the density in the ambient). Furthermore, the numerically calculated time scale for the plume to reach its maximum height of rise, $t_s = 3.04$ s, agrees very well with that predicted by the separable solution, $t_s = 3.14$ s.

5. Conclusions

We demonstrated in § 2 that an analytical series solution can be constructed for continuous sources of buoyancy or momentum in a uniformly stratified environment, and that using only a few terms of the series yields excellent predictions for both the vertical properties and the maximum rise height of the plume fluid. If the source conditions weaken with time, it is also possible to establish the time at which the plume fluid stalls, i.e. $t_s = \pi/N$ (for a pure plume source) or $t_s = \pi/(2N)$ (for a pure jet source). As in the unstratified case (S06), the reduction of driving source conditions for either a plume or a jet results in a narrowing region propagating up the plume or jet with characteristic radius given by $b = 2\alpha z/3$, associated with the time-dependent separable solution.

Reducing the source buoyancy flux (or momentum flux) inevitably reduces the maximum height of rise. However, since the flow is attracted to a separable solution, the time of collapse can be identified and predicted by the separable solution. When the decrease in buoyancy flux is instantaneous, as in § 3, it appears that the plume fluid loses its buoyancy over a finite region after a time $\pi/(2N)$, i.e. precisely half the time until collapse. In the second half of its temporal evolution, $\pi/(2N) < t < \pi/N$, the plume fluid becomes dense compared to its environment and loses its momentum due to the adverse effect of the buoyancy force until collapse occurs at $t = \pi/N$. This extended region of neutrally buoyant fluid is a novel feature of such time-dependent flows and suggests both a finite time and height of rise of a buoyant plume in a stratified environment.

Although, for simplicity, in this paper we concentrated on instantaneous reductions in either source buoyancy or source momentum fluxes and constant buoyancy frequency the governing system of equations allows consideration of more general situations. When the time scale over which the source conditions are reduced is sufficiently small compared to the stall time, t_s, we have found very similar behaviour. However, the dynamics of the flow are much more complex when these two time scales are comparable as the different regions of the flow appear to be more strongly coupled. Nevertheless, the stall time $t_s = \pi/N$ is still a useful predictor for collapse at some (typically relatively high) height within the plume.

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